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Chapter 8
Some Types of HyperNeutrosophic Set (6): MultiNeutrosophic Set and Refined Neutrosophic Set

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Abstract

This paper builds on the foundational advancements introduced in [22, 29–32]. The Neutrosophic Set provides a flexible mathematical framework for managing uncertainty by utilizing three membership functions: truth, indeterminacy, and falsity. Recent extensions, such as the HyperNeutrosophic Set and the SuperHyperNeutrosophic Set, have been developed to address increasingly complex and multidimensional challenges. Comprehensive formal definitions of these concepts are provided in [26].

In this paper, we further extend various specialized classes of Neutrosophic Sets. Specifically, we explore extensions of the MultiNeutrosophic Set and the Refined Neutrosophic Set using HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets, providing detailed analysis and examples.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section presents the foundational concepts and definitions necessary for the discussions in this paper.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

In addressing uncertainty, vagueness, and imprecision in decision-making, various set-theoretic models have been proposed. Among these, Fuzzy Sets introduced by Zadeh [60–66] provide a foundation for handling partial membership. Intuitionistic Fuzzy Sets, developed extensively by Atanassov [8–13], incorporate both membership and non-membership functions for better representation of uncertainty. Similarly, Vague Sets have been explored as a means to model imprecise data [1, 35, 40].

Hyperfuzzy Sets, a generalization of Fuzzy Sets, enable a broader representation of membership by considering subsets of the interval $[0, 1]$ [14, 21, 28, 38, 39]. These models provide enhanced flexibility in handling complex data scenarios.

Neutrosophic Sets, introduced by Smarandache, extend the Fuzzy Set framework by incorporating an indeterminacy component alongside truth and falsity [24, 33, 34, 47–50, 53, 57]. This approach allows for a richer characterization of uncertainty, making it particularly useful in complex decision-making contexts. Advanced studies have further refined Neutrosophic Sets, resulting in the development of HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets, which address high-dimensional and intricate problem domains [23, 26].

The following sections provide definitions and illustrative examples of these concepts, demonstrating their applicability and generalization potential.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [25, 44] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Example 1.3 (Powerset)**.** Let $S = \{a, b\}$. The powerset $\mathcal{P}(S)$ is the set of all subsets of S, including the empty set and S itself.

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Definition 1.4 (n-th Powerset)**.** (cf. [20, 25, 27, 46, 56])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Example 1.5 (First Iteration ($P_1(S)$))**.** By definition, $P_1(S) = \mathcal{P}(S)$. Therefore:

$$P_1(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Example 1.6 (Second Powerset ($P_2(S)$))**.** The second powerset $P_2(S)$ is the powerset of $P_1(S)$. Since $P_1(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, we compute $\mathcal{P}(P_1(S))$:

$$P_2(S) = \mathcal{P}(P_1(S)) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \ldots, P_1(S)\}.$$

Example 1.7 (Third Powerset ($P_3(S)$))**.** The third powerset $P_3(S)$ is obtained by applying the powerset operation to $P_2(S)$:

$$P_3(S) = \mathcal{P}(P_2(S)).$$

Since $P_2(S)$ is a much larger set, $P_3(S)$ contains subsets of $P_2(S)$, including higher-order subsets.

Definition 1.8 (Neutrosophic Set)**.** [47, 48] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 1.9 (Neutrosophic Set in Customer Feedback Analysis)**.** *Scenario:* Evaluating customer satisfaction regarding a newly launched product (cf. [17]).

Example: Let $X = \{\text{Customer 1}, \text{Customer 2}, \text{Customer 3}\}$, representing a set of customers who provided feedback on the product. The membership functions represent their positive feedback (T), uncertainty (I), and negative feedback (F) as follows:

- For Customer 1: $T_A(\text{Customer 1}) = 0.7$ (70% positive feedback), $I_A(\text{Customer 1}) = 0.2$ (20% uncertainty), $F_A(\text{Customer 1}) = 0.1$ (10% negative feedback).
- For Customer 2: $T_A(\text{Customer 2}) = 0.6$, $I_A(\text{Customer 2}) = 0.1$, $F_A(\text{Customer 2}) = 0.3$.
- For Customer 3: $T_A(\text{Customer 3}) = 0.4$, $I_A(\text{Customer 3}) = 0.4$, $F_A(\text{Customer 3}) = 0.2$.

Interpretation:

- Customer 1 shows strong positive feedback with minimal uncertainty and negative feedback.
- Customer 2 provides moderate positive feedback but has a significant level of dissatisfaction ($F_A(\text{Customer 2}) = 0.3$).

- Customer 3 exhibits high uncertainty ($I_A(\text{Customer 3}) = 0.4$), indicating indecisiveness regarding their opinion of the product.

This example demonstrates how a Neutrosophic Set can be used to model customer feedback, capturing not only their satisfaction levels but also their uncertainties and dissatisfaction, enabling a more comprehensive understanding of customer opinions.

Definition 1.10 (HyperNeutrosophic Set)**.** (cf. [21, 23, 26, 28, 54]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Example 1.11 (HyperNeutrosophic Set in Medical Diagnosis)**.** *Scenario:* Evaluating the health status of patients based on multiple diagnostic criteria, incorporating uncertainty and conflicting indicators (cf. [6, 15, 16, 19, 37, 41, 59]).

Let $X = \{\text{Patient 1, Patient 2}\}$, where each patient is assessed using a set of neutrosophic triplets representing degrees of health status (T), uncertainty in diagnosis (I), and severity of illness (F). The evaluations from multiple doctors or diagnostic tests are aggregated as follows:

- For Patient 1:
$$\tilde{\mu}(\text{Patient 1}) = \{(0.9, 0.05, 0.05), (0.8, 0.1, 0.1)\},$$
indicating that one evaluation suggests high health status ($T = 0.9$) with minimal uncertainty ($I = 0.05$), while another is slightly less confident ($T = 0.8, I = 0.1$).
- For Patient 2:
$$\tilde{\mu}(\text{Patient 2}) = \{(0.4, 0.4, 0.2), (0.3, 0.5, 0.2)\},$$
reflecting more uncertainty in diagnosis ($I = 0.4, 0.5$) and lower health status ($T = 0.4, 0.3$).

Interpretation:

- For Patient 1, the aggregated evaluations suggest a strong likelihood of good health, with very low uncertainty and minimal severity of illness.
- For Patient 2, the evaluations highlight significant uncertainty in the diagnosis, coupled with moderately low health status and moderate severity of illness.

This use of HyperNeutrosophic Sets allows for nuanced analysis in medical diagnosis, accommodating varying degrees of confidence and conflicting information from different sources or diagnostic tools.

Definition 1.12 (n-SuperHyperNeutrosophic Set)**.** (cf. [21, 23, 26, 28]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,
$$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$
representing the k-th nested family of non-empty subsets of X.

- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

Example 1.13 (n-SuperHyperNeutrosophic Set)**.** *Scenario:* Multi-tiered analysis of supply chain reliability and uncertainty in a global logistics network.

Example: Let $X = \{\text{Production, Transportation, Warehousing}\}$, representing key components of a global supply chain. We consider a four-level hierarchy:

- *Level 1:* Global regions, e.g., {North America, Europe, Asia}.
- *Level 2:* Countries within regions, e.g., {USA, Germany, Japan}.
- *Level 3:* Distribution centers within countries, e.g., {Center A, Center B, Center C}.
- *Level 4:* Individual suppliers or transport hubs within distribution centers, e.g., {Supplier X, Supplier Y, Hub Z}.

For each level, the n-SuperHyperNeutrosophic Set assigns a family of subsets with neutrosophic membership triplets. For instance:

$$\tilde{A}_4(\text{Supplier X}) = \{(0.9, 0.05, 0.05), (0.85, 0.1, 0.05)\},$$

where each triplet represents:

- T: The reliability of the supplier, indicating truth in meeting delivery schedules.
- I: Uncertainty in performance due to external factors like weather or policy changes.
- F: The failure rate of the supplier in fulfilling commitments.

Interpretation:

- At the global region level, e.g., $\tilde{A}_1(\text{North America}) = \{(0.7, 0.2, 0.1), (0.75, 0.15, 0.1)\}$, reflects broader uncertainties like trade policies or labor strikes.
- At the country level, e.g., $\tilde{A}_2(\text{USA}) = \{(0.8, 0.1, 0.1), (0.85, 0.05, 0.1)\}$, captures national factors like infrastructure reliability or local regulations.
- At the distribution center level, e.g., $\tilde{A}_3(\text{Center A}) = \{(0.85, 0.1, 0.05), (0.8, 0.15, 0.05)\}$, focuses on facility-specific risks.

This hierarchical approach enables granular and comprehensive evaluation of supply chain reliability, accounting for uncertainty and risk at multiple levels of the network.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 m-Valued Refined Neutrosophic set

An m-Valued Refined Neutrosophic Set assigns m-refined truth, indeterminacy, and falsity membership values to elements, capturing granular uncertainty [4–6, 18, 36, 51, 52].

Definition 2.1 (m-Valued Refined Neutrosophic Set)**.** [51] An *m-Valued Refined Neutrosophic Set (m-VRNS)* is defined as follows:

Let U be a universe of discourse. An *m-Valued Refined Neutrosophic Set* N is represented as:

$$N = \{(x, \langle T_x, I_x, F_x \rangle) \mid x \in U\},$$

where:

- $T_x = \{T_x^1, T_x^2, \ldots, T_x^p\}$ is the set of refined truth-membership degrees for x,
- $I_x = \{I_x^1, I_x^2, \ldots, I_x^q\}$ is the set of refined indeterminacy-membership degrees for x,
- $F_x = \{F_x^1, F_x^2, \ldots, F_x^r\}$ is the set of refined falsity-membership degrees for x,
- $T_x^i, I_x^j, F_x^k \in [0, 1] \quad \forall i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}, k \in \{1, \ldots, r\}$,
- $p + q + r = m$, where m represents the total number of refined components.

The following condition must hold for each $x \in U$:

$$0 \leq \sum_{i=1}^{p} T_x^i + \sum_{j=1}^{q} I_x^j + \sum_{k=1}^{r} F_x^k \leq m.$$

Notes:

1. The sets T_x, I_x, F_x can represent different types of truth, indeterminacy, and falsity degrees (e.g., based on multiple criteria or perspectives).
2. The upper bound m ensures that the sum of all degrees (truth, indeterminacy, falsity) does not exceed the total refined capacity.
3. The model generalizes the classical neutrosophic set by allowing finer granularity in the representation of uncertainty.

Example 2.2 (m-Valued Refined Neutrosophic Set)**.** *Scenario:* Evaluating student performance in a multi-criteria assessment.

Let $U = \{\text{Alice, Bob, Charlie}\}$, where U represents students under evaluation. Each student is assessed on three main criteria: knowledge, teamwork, and creativity. The truth (T), indeterminacy (I), and falsity (F) degrees are refined into sub-components to represent their performance in finer granularity.

- For Alice:
$$T_{\text{Alice}} = \{0.8, 0.7\}, \quad I_{\text{Alice}} = \{0.1\}, \quad F_{\text{Alice}} = \{0.1, 0.2\},$$
where:
  - T_{Alice}: High scores in knowledge (0.8) and teamwork (0.7).
  - I_{Alice}: Slight uncertainty (0.1) due to variable creativity.
  - F_{Alice}: Weak performance in advanced tasks (0.1) and group leadership (0.2).

  Total $m = 5$.

- For Bob:
$$T_{\text{Bob}} = \{0.6, 0.5\}, \quad I_{\text{Bob}} = \{0.2, 0.3\}, \quad F_{\text{Bob}} = \{0.4\},$$
where:
  - T_{Bob}: Moderate scores in creativity (0.6) and teamwork (0.5).
  - I_{Bob}: Higher uncertainty (0.2, 0.3) due to inconsistent knowledge application.
  - F_{Bob}: Weak performance in collaboration (0.4).

  Total $m = 6$.
- For Charlie:
$$T_{\text{Charlie}} = \{0.7, 0.6\}, \quad I_{\text{Charlie}} = \{0.2\}, \quad F_{\text{Charlie}} = \{0.3, 0.2\},$$
where:
  - T_{Charlie}: Strong knowledge (0.7) and decent creativity (0.6).
  - I_{Charlie}: Moderate uncertainty (0.2) due to inconsistent teamwork.
  - F_{Charlie}: Weak performance in leadership (0.3) and practical implementation (0.2).

  Total $m = 5$.

Interpretation: Each student is evaluated on finer criteria, allowing for a nuanced assessment of their performance across different dimensions. This granularity is achieved using the m-Valued Refined Neutrosophic Set framework, where the truth, indeterminacy, and falsity degrees reflect detailed attributes under evaluation.

Definition 2.3 (m-Valued Refined HyperNeutrosophic Set (m-VRHNS))**.** Let U be a non-empty universe, and let
$$\mathcal{R}_m = \left\{ (T, I, F) \;\middle|\; T, I, F \subseteq [0,1],\ |T| + |I| + |F| = m,\ \sum T + \sum I + \sum F \leq m \right\}$$
be the family of all *refined* triplets of sets representing truth, indeterminacy, and falsity degrees, subject to the total m constraint. Let $\widetilde{P}(\mathcal{R}_m)$ be the family of non-empty subsets of $\mathcal{R}_m$. A *m-Valued Refined HyperNeutrosophic Set (m-VRHNS)* $\widetilde{N}$ is a mapping
$$\widetilde{N} : U \longrightarrow \widetilde{P}(\mathcal{R}_m),$$
where for each $x \in U$, $\widetilde{N}(x) \subseteq \mathcal{R}_m$ is a *set* of refined triplets $(T_x^\alpha, I_x^\alpha, F_x^\alpha)$ with α indexing different possible combinations, each obeying
$$\sum (T_x^\alpha) + \sum (I_x^\alpha) + \sum (F_x^\alpha) \leq m.$$

Interpretation:

- If $\widetilde{N}(x)$ is restricted to *one* triple (T_x, I_x, F_x), we get an *m-Valued Refined Neutrosophic Set* (Definition **??**).
- If $m = 3$ and each set is single-valued, we revert to a standard hyperneutrosophic set in $[0,1]^3$.

Theorem 2.4. *Every m-Valued Refined Neutrosophic Set is a special case of an m-Valued Refined HyperNeutrosophic Set.*

Proof. Let N be an m-Valued Refined Neutrosophic Set, with each $x \in U$ having one triple $(T_x, I_x, F_x) \in \mathcal{R}_m$. We define a mapping $\widetilde{N}$ by
$$\widetilde{N}(x) = \{ (T_x, I_x, F_x) \}.$$
Hence each x has a singleton set of refined triplets. By construction, (T_x, I_x, F_x) obeys $\sum T_x + \sum I_x + \sum F_x \leq m$. Therefore N is embedded into $\widetilde{N}$ as a degenerate (single triple) membership approach. □

Theorem 2.5. *By letting $m = 3$ and restricting each T, I, F to a single value in $[0,1]$, we revert to an ordinary HyperNeutrosophic Set in $[0,1]^3$.*

Proof. A *HyperNeutrosophic Set* $\tilde{\mu} : U \to \mathcal{P}([0,1]^3)$ assigns each x a set of triplets (T, I, F) with $T+I+F \leq 3$. In the refined approach, $m = 3$ means we distribute the total 3 among T, I, F sets. If we force each set $T = \{t\}, I = \{i\}, F = \{f\}$, each containing exactly one numeric value, we get a single triplet $(t, i, f) \in [0,1]^3$, as usual. By allowing a set of such singletons, we reproduce the standard hyperneutrosophic membership in $[0,1]^3$. □

Definition 2.6 (m-Valued Refined n-SuperHyperNeutrosophic Set (m-VRHNS$_n$))**.** Let U be a non-empty universe, $n \geq 0$ an integer, and m a positive integer for refined neutrosophic components. Define $\mathcal{R}_m$ as in Definition 2.3 and $\widetilde{P}(\mathcal{R}_m)$ as the family of non-empty subsets of $\mathcal{R}_m$. Let $\widetilde{\mathcal{P}}_n^*(U)$ be the n-th power set hierarchy of U minus the empty set. An *m-Valued Refined n-SuperHyperNeutrosophic Set (m-VRHNS$_n$)* is a mapping:

$$\widetilde{N}_n : \widetilde{\mathcal{P}}_n^*(U) \longrightarrow \widetilde{P}(\mathcal{R}_m),$$

where each $A \in \widetilde{\mathcal{P}}_n^*(U)$ is assigned a *set* of refined neutrosophic triplets (T_A, I_A, F_A) with $|T_A|+|I_A|+|F_A| = m$, each satisfying

$$\sum T_A + \sum I_A + \sum F_A \leq m.$$

Theorem 2.7. *Every m-Valued Refined HyperNeutrosophic Set is a special case of an m-Valued Refined n-SuperHyperNeutrosophic Set for $n = 0$ or 1.*

Proof. Let $\widetilde{N}$ be an m-Valued Refined HyperNeutrosophic Set: $U \to \widetilde{P}(\mathcal{R}_m)$. In Definition 2.6, if $n = 0$, $\widetilde{\mathcal{P}}_0^*(U) = U$, so

$$\widetilde{N}_0 : U \to \widetilde{P}(\mathcal{R}_m)$$

matches $\widetilde{N}$. Alternatively, if $n = 1$, define

$$\widetilde{N}_1(\{x\}) := \widetilde{N}(x), \quad \widetilde{N}_1(A) = \varnothing \text{ if } A \neq \{x\}.$$

Hence the single-level m-VRHNS is embedded in an n-super environment as a special case. □

Theorem 2.8. *By letting $m = 3$ and forcing each (T_A, I_A, F_A) to be singletons in $[0,1]$, we revert to a standard n-SuperHyperNeutrosophic set in numeric membership $[0,1]^3$.*

Proof. An *n-SuperHyperNeutrosophic Set* $\tilde{\mu}_n$ maps $\widetilde{\mathcal{P}}_n^*(U)$ to sets of triplets $(T, I, F) \in [0,1]^3$ with $T+I+F \leq 3$. In the m-refined approach, $m = 3$ means we distribute the total across T, I, F. If each $T = \{t\}, I = \{i\}, F = \{f\}$ is a single numeric value, we replicate ordinary triplets. By allowing sets of such singletons, we replicate the standard n-SuperHyperNeutrosophic membership. □

2.2 MultiNeutrosophic Set

The concept of MultiNeutrosophic Sets has been extensively studied in several research papers [2, 3, 7, 55]. Related concepts, such as the Refined Fuzzy Set, are also well-documented in the literature [42, 43, 45, 58]. We extend the framework of MultiNeutrosophic Sets by incorporating HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets. The formal definitions and associated concepts are provided below.

Definition 2.9 (MultiNeutrosophic Set)**.** (cf. [55]) Let $\mathcal{U}$ be a universe of discourse, and let M be a subset of $\mathcal{U}$. A *MultiNeutrosophic Set (MNS) M* is defined as:

$$M = \left\{ \left(x, \langle T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s \rangle\right) \,\middle|\, x \in \mathcal{U} \right\},$$

where:

- $p, r, s \geq 0$ with $p + r + s = n \geq 2$,
- At least one of p, r, s satisfies ≥ 2 to ensure the multiplicity of truth (T), indeterminacy (I), or falsehood (F),
- $T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s \subseteq [0,1]$,

- The following condition is satisfied:

$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n.$$

Example 2.10 (Application of MultiNeutrosophic Set)**.** *Scenario:* Evaluating job applicants by multiple experts on their qualifications, experience, and cultural fit.

Example: Let $\mathcal{U} = \{\text{Alice, Bob, Charlie}\}$, representing the candidates. Each candidate is evaluated on three aspects by multiple experts:

- Truth (T): Positive qualifications and relevant skills,
- Indeterminacy (I): Ambiguity in the applicant's experience or behavior,
- Falsehood (F): Negative feedback or lack of required competencies.

The evaluations are provided by $n = 3 + 2 + 4 = 9$ sources as follows:

- For Alice:

$$T_{\text{Alice}} = \{0.8, 0.7, 0.6\}, \quad I_{\text{Alice}} = \{0.2, 0.3\}, \quad F_{\text{Alice}} = \{0.4, 0.3, 0.5, 0.2\}.$$

- For Bob:

$$T_{\text{Bob}} = \{0.7, 0.6, 0.5\}, \quad I_{\text{Bob}} = \{0.4, 0.3\}, \quad F_{\text{Bob}} = \{0.5, 0.4, 0.3, 0.2\}.$$

- For Charlie:

$$T_{\text{Charlie}} = \{0.5, 0.6, 0.4\}, \quad I_{\text{Charlie}} = \{0.1, 0.2\}, \quad F_{\text{Charlie}} = \{0.6, 0.5, 0.4, 0.3\}.$$

Each candidate is evaluated on multiple dimensions (truth, indeterminacy, and falsehood) by different experts. This structure enables nuanced decision-making by aggregating diverse perspectives. For instance, Alice's high truth values $(0.8, 0.7, 0.6)$ indicate strong qualifications, while her indeterminacy and falsehood values highlight specific areas of uncertainty or weakness.

Definition 2.11 (MultiHyperNeutrosophic Set)**.** Let $\mathcal{U}$ be a universe of discourse, and let M be a subset of $\mathcal{U}$. A *MultiHyperNeutrosophic Set (MHNS) M* is defined as:

$$M = \{(x, \tilde{\mu}(x)) \mid x \in \mathcal{U}\},$$

where $\tilde{\mu}(x) \subseteq \mathcal{P}([0,1]^3)$ is a set of neutrosophic membership triplets:

$$\tilde{\mu}(x) = \left\{(T_j, I_k, F_l) \;\middle|\; j = 1, \ldots, p;\ k = 1, \ldots, r;\ l = 1, \ldots, s\right\}.$$

The following conditions must be satisfied:

- $T_j, I_k, F_l \subseteq [0,1]$, where T_j, I_k, and F_l represent subsets of truth, indeterminacy, and falsehood degrees, respectively.
- The parameters $p, r, s \geq 1$ satisfy $p + r + s = n \geq 2$, ensuring at least one component has multiplicity ≥ 2.
- For all $x \in \mathcal{U}$, the following condition holds:

$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n.$$

Example 2.12 (Example: MultiHyperNeutrosophic Set in Team Evaluation)**.** *Scenario:* A company is evaluating project teams based on performance, innovation, and collaboration. Feedback is provided by multiple experts.

Example: Let $\mathcal{U} = \{\text{Team A, Team B, Team C}\}$, representing the project teams. For each team:

- Truth (T): Measures team performance, such as task completion and success rates.
- Indeterminacy (I): Reflects ambiguity in team processes or communication.
- Falsehood (F): Captures team failures or conflicts.

Evaluations by $p = 3$, $r = 2$, $s = 4$ sources are as follows:

- For Team A:
$$T_{\text{Team A}} = \{[0.8, 0.9], [0.7, 0.8], [0.6, 0.7]\},$$
$$I_{\text{Team A}} = \{[0.2, 0.3], [0.1, 0.2]\},$$
$$F_{\text{Team A}} = \{[0.1, 0.2], [0.3, 0.4], [0.2, 0.3], [0.4, 0.5]\}.$$
- For Team B:
$$T_{\text{Team B}} = \{[0.7, 0.8], [0.6, 0.7], [0.5, 0.6]\},$$
$$I_{\text{Team B}} = \{[0.3, 0.4], [0.2, 0.3]\},$$
$$F_{\text{Team B}} = \{[0.4, 0.5], [0.3, 0.4], [0.2, 0.3], [0.5, 0.6]\}.$$
- For Team C:
$$T_{\text{Team C}} = \{[0.6, 0.7], [0.5, 0.6], [0.4, 0.5]\},$$
$$I_{\text{Team C}} = \{[0.1, 0.2], [0.0, 0.1]\},$$
$$F_{\text{Team C}} = \{[0.5, 0.6], [0.4, 0.5], [0.3, 0.4], [0.6, 0.7]\}.$$

The evaluations provide a comprehensive view of each team's strengths (T), uncertainties (I), and weaknesses (F), enabling informed decision-making for resource allocation and performance improvement.

Definition 2.13 (Multi n-SuperHyperNeutrosophic Set)**.** Let $\mathcal{U}$ be a universe of discourse. A *Multi n-SuperHyperNeutrosophic Set (Multi n-SHNS)* is a recursive generalization of MultiHyperNeutrosophic Sets, defined as:
$$\tilde{M}_n : \mathcal{P}_n(\mathcal{U}) \to \mathcal{P}_n([0, 1]^3),$$
where:

- $\mathcal{P}_1(\mathcal{U}) = \mathcal{P}(\mathcal{U})$, the power set of $\mathcal{U}$,
- For $k \geq 2$,
$$\mathcal{P}_k(\mathcal{U}) = \mathcal{P}(\mathcal{P}_{k-1}(\mathcal{U})),$$
representing the k-th nested family of non-empty subsets of $\mathcal{U}$.
- For each $A \in \mathcal{P}_n(\mathcal{U})$, $\tilde{M}_n(A) \subseteq \mathcal{P}_n([0, 1]^3)$ is a family of subsets of neutrosophic triplets (T, I, F) satisfying:
$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n.$$

Theorem 2.14. *A Multi n-SuperHyperNeutrosophic Set generalizes the concepts of MultiNeutrosophic Sets, HyperNeutrosophic Sets, and n-SuperHyperNeutrosophic Sets.*

Proof. 1. When $p = r = s = 1$, the structure reduces to a MultiNeutrosophic Set, as each evaluation is single-valued.

2. When $n = 1$, the structure aligns with the HyperNeutrosophic Set, mapping elements directly to neutrosophic triplets.

3. For $n > 1$, the recursive construction introduces hierarchical relationships, extending n-SuperHyperNeutrosophic Sets to multi-source contexts.

Thus, the Multi n-SuperHyperNeutrosophic Set encompasses all these frameworks. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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